

One-shot source coding with coded side information available at the decoder

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Abstract—One-shot achievable rate region for source coding when coded side information is available at the decoder (source coding with a helper) is proposed. The achievable region proposed is in terms of conditional smooth max Rényi entropy and smooth max Rényi divergence. Asymptotically (in the limit of large block lengths) this region is quantified in terms of spectral-sup conditional entropy rate and spectral-sup mutual information rate. In particular, it coincides with the rate region derived in the limit of unlimited independent and identically distributed copies of the sources.

I. INTRODUCTION

The derivation of most of the fundamental results in information theory relies on the assumption that a random experiment is repeated identically and independently for large enough time. However, in practical scenarios both of these assumptions are not always justifiable. To overcome the limitations posed by these assumptions Renner et al. introduced the notion of *one-shot* information theory. One-shot information theory relies on the fact that a random experiment is available only *once*. Thus removing both the assumptions together.

The first one-shot bounds were given for the task of one-shot source coding [1]. These bounds were based on smooth Rényi entropies. The notion of smooth Rényi entropies were introduced for the very first time in the same work, i.e., in Ref. [1]. The elegance of the one-shot bounds obtained in Ref. [1] is that these bounds coincide with the Shannon entropy [2] of the information source in the limit of unlimited independent and identically distributed (i.i.d.) copies of the source. Furthermore, these bounds coincide with spectral sup-entropy rate as defined by Han and Verdú in Ref. [3] in the limit of unlimited arbitrarily distributed copies of the source. One-shot bounds for distributed source coding were given by Sharma et al. in [4]. In [5] Wang et al. gave one-shot bounds for the channel coding problem in terms of smooth min Rényi divergence.

There has been a considerable work on the one-shot bounds for the quantum case under various scenarios (see for example Refs. [6], [7], [8], [9], [10], [11] and references therein).

In this work we give one-shot achievable rate region for source coding when coded state side information is available at the decoder. The achievable rate region derived for this problem is in terms of smooth max Rényi divergence and conditional smooth max Rényi entropy. The notion of smooth max Rényi divergence was introduced by Datta for the quantum case in [12]. We further show that the achievable region obtained asymptotically coincides with the rate region derived in [13].

The rest of this paper is organized as follows. In Section II we discuss the notations which we will be using throughout this paper. In Section III we give the definitions of smooth conditional Rényi entropy of order zero and smooth max Rényi divergence. We then prove two lemmas pertaining to the properties of smooth max Rényi divergence. Although, the proof of Lemma 3 is known in the quantum case we give a totally different proof. In particular, our proof involves more straightforward arguments. In Section IV we state and prove the achievable region for source coding problem when coded side information is available at the decoder.

II. NOTATIONS

In the discussions below we will be using X to represent a random variable. We will assume that all the random variables are discrete and have finite range. We represent a random sequence of length n by X^n and a particular realization of X^n by \mathbf{x} . Notation \mathbf{X} will be used to represent an arbitrary sequence of random variables, i.e., $\mathbf{X} = \{X_n\}_{n=1}^{\infty}$. We use the notation $|\cdot|$ to represent the cardinality of a set. The set $\{\mathbf{x} : P_{X^n}(\mathbf{x}) > 0\}$ is denoted by $\text{Supp}(P_{X^n})$. We use the notation

$$X \rightarrow Y \rightarrow Z$$

to denote the fact that random variables X , Y and Z form a Markov chain. We represent the following set of real numbers

$$\{x : 0 \leq x < \infty\}$$

by \mathbb{R}^+ . $\mathcal{X} \times \mathcal{Y}$ will represent the cartesian product of two sets. Similarly $(\mathcal{X} \times \mathcal{Y})^n$ will represent the n -th

Cartesian product of the set $\mathcal{X} \times \mathcal{Y}$. The notation \mathbb{N} is used to represent the set of natural numbers. Throughout this paper we will assume that \log is to the base 2.

III. SMOOTH RÉNYI DIVERGENCE OF ORDER INFINITY AND CONDITIONAL SMOOTH RÉNYI ENTROPY OF ORDER ZERO

Definition 1: (Max Rényi entropy [14]) Let $X \sim P_X$, with range \mathcal{X} . The zero order Rényi entropy of X is defined as

$$H_0(X) := \log \text{Supp}(P_X).$$

Definition 2: (Conditional smooth max Rényi entropy [15]) Let $(X, Y) \sim P_{XY}$, with range $\mathcal{X} \times \mathcal{Y}$. For $\varepsilon \geq 0$, the conditional smooth Rényi entropy of order zero of X given Y is defined as

$$H_0^\varepsilon(X|Y) := \min_{Q \in \mathcal{B}^\varepsilon(P_{XY})} \log \max_{y \in \mathcal{Y}} |\text{Supp}(Q(X|Y=y))|,$$

where $\mathcal{B}^\varepsilon(P_{XY}) = \{Q : \sum_{x,y \in \mathcal{X} \times \mathcal{Y}} Q(x,y) \geq 1 - \varepsilon, \forall (x,y) \in \mathcal{X} \times \mathcal{Y}, P_{XY}(x,y) \geq Q(x,y) \geq 0\}$ and $Q(X=x|Y=y) := \frac{Q(x,y)}{P_Y(y)}$, for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. With the convention that $Q(X=x|Y=y) := 0$ if $P_Y(y) = 0$.

Definition 3: (Max Rényi divergence [14]) Let P and Q be two probability mass functions on the set \mathcal{X} such that $\text{Supp}(P) \subseteq \text{Supp}(Q)$. The max Rényi divergence between P and Q is defined as

$$D_\infty(P||Q) := \log \max_{x: P(x) > 0} \frac{P(x)}{Q(x)}.$$

Definition 4: (Smooth max Rényi divergence) Let P and Q be two probability mass functions on the set \mathcal{X} such that $\text{Supp}(P) \subseteq \text{Supp}(Q)$. The smooth max Rényi divergence between P and Q for $\varepsilon \in [0, 1)$ is defined as

$$D_\infty^\varepsilon(P||Q) := \log \inf_{\phi \in \mathcal{B}^\varepsilon(P)} \max_{x: P(x) > 0} \frac{\phi(x)}{Q(x)},$$

where

$$\mathcal{B}^\varepsilon(P) = \left\{ \phi : 0 \leq \phi(x) \leq P(x), \forall x \in \mathcal{X} \text{ and } \sum_{x \in \mathcal{X}} \phi(x) \geq 1 - \varepsilon \right\}.$$

Notice that $D_\infty^\varepsilon(P||Q)$ is a non increasing function of ε .

Lemma 1: (Datta and Renner [6]) Let $(\mathbf{X}, \mathbf{Y}) = \{(X_n, Y_n)\}_{n=1}^\infty$ be an arbitrary random sequence taking values over the set $\{(\mathcal{X} \times \mathcal{Y})^n\}_{n=1}^\infty$, where $(\mathcal{X} \times \mathcal{Y})^n$ is the n -th Cartesian product of $\mathcal{X} \times \mathcal{Y}$. Then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{H_0^\varepsilon(X^n|Y^n)}{n} = \overline{H}(\mathbf{X}|\mathbf{Y}),$$

where

$$\overline{H}(\mathbf{X}|\mathbf{Y}) := \inf \left\{ \alpha \mid \liminf_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{X^n|Y^n}(X^n|Y^n)} \leq \alpha \right\} = 1 \right\}.$$

$\overline{H}(\mathbf{X}|\mathbf{Y})$ is called the spectral-sup conditional entropy rate of \mathbf{X} given \mathbf{Y} [16]. In particular, if $(\mathbf{X}, \mathbf{Y}) = \{(X_n, Y_n)\}_{n=1}^\infty$ is a random sequence of independent and identically distributed random pairs distributed according to P_{XY} then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{H_0^\varepsilon(X^n|Y^n)}{n} = H(X|Y).$$

Lemma 2: Let P and Q be two probability mass functions defined on the set \mathcal{X} , where $\text{Supp}(P) \subseteq \text{Supp}(Q)$ and $|\mathcal{X}| < \infty$. There exists $\phi \in \mathcal{B}^\varepsilon(P)$ such that

$$D_\infty^\varepsilon(P||Q) = \log \max_{x \in \mathcal{X}} \frac{\phi(x)}{Q(x)}. \quad (1)$$

Proof: Without loss of generality we assume here that $\mathcal{X} \subset \mathbb{N}$. We construct the ϕ_i s by decreasing the P_i s such that the total decrease is ε , i.e., $\sum_{i \in \mathcal{X}} (P_i - \phi_i) = \varepsilon$. The following procedure achieves the above.

Step 0 \rightarrow (Initialization) $\phi_i = P_i \forall i \in \mathcal{X}$.

Step 1 \rightarrow Let r_1, r_2 be the largest and the second largest ratios in

$$\mathcal{A} = \left\{ \frac{\phi_i}{q_i} : i \in \mathcal{X} \right\} \quad (2)$$

respectively. Let I be the collection of all i s that have the highest ratio, i.e.,

$$I = \left\{ i \in \mathcal{X} : \frac{\phi_i}{Q_i} = r_1 \right\}. \quad (3)$$

If $|\mathcal{A}| = 1$ then notice that $I = \mathcal{X}$. In this case start decreasing all ϕ_i s where $i \in \mathcal{X}$ such that all the indices continue to have constant ratio, i.e., $\frac{\phi_i}{Q_i} = \frac{\phi_j}{Q_j} \forall i, j \in \mathcal{X}$. Continue this process until we run out of ε , i.e., $\sum_{i \in \mathcal{X}} (P_i - \phi_i) = \varepsilon$ in which case end the procedure. Else go to step 2.

Step 2 \rightarrow We start decreasing all ϕ_i s where $i \in I$ such that indices in I continue to have the highest ratio, i.e., $\frac{\phi_i}{Q_i} = \frac{\phi_j}{Q_j} \forall i, j \in I$. As a result, r_1 will start decreasing. Continue decreasing till either

Case 1: r_1 hits r_2 , i.e., $r_1 = r_2$ in which case stop decreasing any further. Goto step 1. Or

Case 2: we run out of ε , i.e., $\sum_{i \in I} (P_i - \phi_i) = \varepsilon$ in which case end the procedure.

We claim that the ϕ constructed by the above procedure is such that

$$\log \max_{x \in \mathcal{X}} \frac{\phi(x)}{Q(x)} = D_\infty^\varepsilon(P||Q). \quad (4)$$

We give a proof by contradiction to prove (25). Let $\phi' \in \mathcal{B}^\varepsilon(P)$ be the output of some other procedure such that

$$\log \max_{x \in \mathcal{X}} \frac{\phi(x)}{Q(x)} > \log \max_{x \in \mathcal{X}} \frac{\phi'(x)}{Q(x)}. \quad (5)$$

Let $\hat{\mathcal{A}} = \{i \in \mathcal{X} : \phi_i < P_i\}$. Notice that for every $i, j \in \hat{\mathcal{A}}$

$$\frac{\phi_i}{Q_i} = \frac{\phi_j}{Q_j}.$$

It is easy to observe that for (5) to hold ϕ' must satisfy the following

$$\phi'_i < \phi_i, \forall i \in \mathcal{X}. \quad (6)$$

However, this is not possible because this new procedure will not have enough ε to accomplish (6), i.e.,

$$\sum_{i \in \hat{\mathcal{A}}} (P_i - \phi'_i) > \varepsilon.$$

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Remark: It is easy to observe from the proof of Lemma 2 that for $\varepsilon \in [0, 1]$,

$$\text{Supp}(\phi) = \text{Supp}(P), \quad (7)$$

where ϕ satisfies (1).

Lemma 3: Let $\mathbf{P} = \{P_n\}_{n=1}^\infty$ and $\mathbf{Q} = \{Q_n\}_{n=1}^\infty$ be an arbitrary sequences of probability mass functions defined on the set $\{\mathcal{X}^n\}_{n=1}^\infty$, where \mathcal{X}^n is the n -th cartesian product of the set \mathcal{X} and $|\mathcal{X}| < \infty$. Assume that for every $n \in \mathbb{N}$, $\text{Supp}(P_n) \subseteq \text{Supp}(Q_n)$. Then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n||Q_n) = \bar{I}(\mathbf{P}; \mathbf{Q}), \quad (8)$$

where

$$\bar{I}(\mathbf{P}; \mathbf{Q}) := \inf \left\{ \alpha \mid \liminf_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n} \log \frac{P_n}{Q_n} \leq \alpha \right\} = 1 \right\}. \quad (9)$$

$\bar{I}(\mathbf{P}; \mathbf{Q})$ is called the spectral sup-mutual information rate between \mathbf{P} and \mathbf{Q} [16]. In particular, if $\mathbf{P} = \{P^{\times n}\}_{n=1}^\infty$ and $\mathbf{Q} = \{Q^{\times n}\}_{n=1}^\infty$, where $P^{\times n}$ and $Q^{\times n}$ represent the product distributions of P and Q on \mathcal{X}^n . Then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n||Q_n) = D(P||Q). \quad (10)$$

Proof: We will first prove

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n||Q_n) \leq \bar{I}(\mathbf{P}; \mathbf{Q}).$$

Consider any $\lambda > \bar{I}(\mathbf{P}; \mathbf{Q})$. Let us define the following set

$$\mathcal{A}_n(\lambda) := \left\{ \mathbf{x} : \frac{1}{n} \log \frac{P_n(\mathbf{x})}{Q_n(\mathbf{x})} \leq \lambda \right\}. \quad (11)$$

Let $\phi_n : \mathcal{X}^n \rightarrow [0, 1]$, $n \in \mathbb{N}$, such that

$$\phi_n(\mathbf{x}) = \begin{cases} P_n(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{A}_n(\lambda), \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

From (9) it easily follows that

$$\lim_{n \rightarrow \infty} \Pr\{\mathcal{A}_n(\lambda)\} = 1. \quad (13)$$

Thus from our construction of ϕ_n , (12), it follows that

$$\lim_{n \rightarrow \infty} \sum_{\mathbf{x} \in \mathcal{X}^n} \phi_n(\mathbf{x}) = \lim_{n \rightarrow \infty} \Pr\{\mathcal{A}_n(\lambda)\} = 1. \quad (14)$$

Using (12) and (14) observe that for n large enough

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n||Q_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \max_{\mathbf{x} \in \mathcal{A}_n(\lambda)} \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})} \stackrel{a}{\leq} \lambda,$$

where a follows from (11) and (12).

We now prove the other side, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n||Q_n) \geq \bar{I}(\mathbf{P}; \mathbf{Q}).$$

Consider any $\gamma < \bar{I}(\mathbf{P}; \mathbf{Q})$. For every $n \in \mathbb{N}$, let us define the following set

$$\mathcal{A}_n(\gamma) := \left\{ \mathbf{x} : \frac{1}{n} \log \frac{P_n(\mathbf{x})}{Q_n(\mathbf{x})} \geq \gamma \right\}. \quad (15)$$

From (9) it follows that there exists $\eta \in (0, 1]$, such that

$$\limsup_{n \rightarrow \infty} \Pr\{\mathcal{A}_n(\gamma)\} = \eta. \quad (16)$$

Since $\Pr\{\mathcal{A}_n(\gamma)\} + \Pr\{\mathcal{A}_n^c(\gamma)\} = 1$, for every $n \in \mathbb{N}$, we have

$$\liminf_{n \rightarrow \infty} \Pr\{\mathcal{A}_n^c(\gamma)\} = 1 - \eta. \quad (17)$$

For every $\varepsilon \in (0, \eta)$, let us define a sequence of positive functions $\{\phi_n\}_{n=1}^\infty$, such that for every $n \in \mathbb{N}$

$$\phi_n : \mathcal{X}^n \rightarrow [0, 1], \phi_n(\mathbf{x}) \leq P_n(\mathbf{x}), \forall \mathbf{x} \in \mathcal{X}^n$$

$$\text{and } \sum_{\mathbf{x} \in \mathcal{X}^n} \phi_n(\mathbf{x}) \geq 1 - \varepsilon. \quad (18)$$

We now claim that for large enough n , $\text{Supp}(\phi_n) \cap \mathcal{A}_n(\gamma) \neq \emptyset$. To prove this claim, suppose that $\text{Supp}(\phi_n) \cap \mathcal{A}_n(\gamma) = \emptyset$. This would further imply that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{\mathbf{x} \in \mathcal{X}^n} \phi_n(\mathbf{x}) &\leq \liminf_{n \rightarrow \infty} \Pr\{\mathcal{A}_n^c(\gamma)\} \\ &\stackrel{a}{=} 1 - \eta \\ &\stackrel{b}{<} 1 - \varepsilon, \end{aligned} \quad (19)$$

where a follows from (17) and b follows because $\varepsilon < \eta$. Notice that (19) contradicts (18).

Thus for n large enough,

$$\begin{aligned} 1 - \varepsilon &\leq \sum_{\mathbf{x} \in \mathcal{A}_n^c(\gamma)} \phi_n(\mathbf{x}) + \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}) \\ &\leq \Pr\{\mathcal{A}_n^c(\gamma)\} + \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}). \end{aligned}$$

By rearranging the terms in the above equation we get

$$1 - \varepsilon - \Pr\{\mathcal{A}_n^c(\gamma)\} \leq \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}). \quad (20)$$

Taking lim sup on both sides of (20), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}) &\geq 1 - \varepsilon - \liminf_{n \rightarrow \infty} \Pr\{\mathcal{A}_n^c(\gamma)\} \\ &\geq \eta - \varepsilon. \end{aligned} \quad (21)$$

(21) follows from (17). Now notice the following set of inequalities for large enough n

$$\begin{aligned} 1 &\geq \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} P_n(\mathbf{x}) \\ &\stackrel{a}{\geq} \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} 2^{n\gamma} Q_n(\mathbf{x}) \\ &\stackrel{b}{\geq} 2^{(n\gamma - \max_{\mathbf{x} \in \mathcal{X}^n} \log \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})})} \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}) \end{aligned} \quad (22)$$

where a follows from (15); b follows from the fact that for every $\mathbf{x} \in \mathcal{A}_n(\gamma)$,

$$\frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})} \leq \max_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})} \leq \max_{\mathbf{x} \in \mathcal{X}^n} \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})}$$

By taking log on both sides of (22) and rearranging the terms we get

$$\max_{\mathbf{x} \in \mathcal{X}^n} \frac{1}{n} \log \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})} \geq \gamma + \frac{1}{n} \log \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}).$$

Taking lim sup on both sides of the above equation we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \max_{\mathbf{x} \in \mathcal{X}^n} \frac{1}{n} \log \frac{\phi_n(\mathbf{x})}{Q_n(\mathbf{x})} &\geq \gamma + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{x} \in \mathcal{A}_n(\gamma)} \phi_n(\mathbf{x}) \\ &\geq \gamma \end{aligned} \quad (23)$$

where (23) follows from (21). Notice that (23) is true for every ϕ_n satisfying (18). Thus

$$\limsup_{n \rightarrow \infty} \frac{1}{n} D_\infty^\varepsilon(P_n \| Q_n) \geq \gamma. \quad (24)$$

Since (24) is true for every $\varepsilon \in (0, \eta)$, the result will hold true for $\varepsilon \downarrow 0$.

(10) easily follows from the law of large numbers and (8). This completes the proof. \blacksquare

IV. SOURCE CODING WITH CODED STATE SIDE INFORMATION AVAILABLE AT THE DECODER

Let $(X^n, Y^n) \sim P_{X^n Y^n}$, with range $(\mathcal{X} \times \mathcal{Y})^n$, where

$$(X^n, Y^n) := [(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)].$$

The n -shot source coding with coded side information available at the decoder is formulated as follows. We first define two sets of integers

$$\mathcal{M}_n^{(1)} = \{1, \dots, 2^{\ell_{\text{d-enc}}^\varepsilon(X^n)}\}, \quad (25)$$

$$\mathcal{M}_n^{(2)} = \{1, \dots, 2^{\ell_{\text{d-enc}}^\varepsilon(Y^n)}\} \quad (26)$$

called the codes. Choose arbitrary mappings $e_n^{(1)} : \mathcal{X}^n \rightarrow \mathcal{M}_n^{(1)}$ (encoder1) and $e_n^{(2)} : \mathcal{Y}^n \rightarrow \mathcal{M}_n^{(2)}$ (encoder2). We call

$$\begin{aligned} \frac{\ell_{\text{d-enc}}^\varepsilon(X^n)}{n} &= \frac{\log |\mathcal{M}_n^{(1)}|}{n}, \\ \frac{\ell_{\text{d-enc}}^\varepsilon(Y^n)}{n} &= \frac{\log |\mathcal{M}_n^{(2)}|}{n} \end{aligned}$$

the coding rates of the encoder 1 and encoder 2, respectively. The decoder $d_n : \mathcal{M}_n^{(1)} \times \mathcal{M}_n^{(2)} \rightarrow \mathcal{X}^n$ receives two outputs $e_n^{(1)}(\mathbf{x})$ and $e_n^{(2)}(\mathbf{y})$ from the two encoders and tries to reconstruct the original source output \mathbf{x} . Thus the probability of error for this task is defined as

$$P_e^n := \Pr\{X^n \neq \hat{X}^n\},$$

where $\hat{X}^n = d_n(e_n^{(1)}(X^n), e_n^{(2)}(Y^n))$. Note here that the encoders $e_n^{(1)}$ and $e_n^{(2)}$ do not cooperate with each other. We call the triplet $(e_n^{(1)}, e_n^{(2)}, d_n)$ of two encoders and one decoder with the two codes in (25) and (26) and the error probability ε the $(n, 2^{\ell_{\text{d-enc}}^\varepsilon(X^n)}, 2^{\ell_{\text{d-enc}}^\varepsilon(Y^n)}, \varepsilon)$ n -shot code.

In this coding system we wish to minimize the two coding rates $\frac{\ell_{\text{d-enc}}^\varepsilon(X^n)}{n}$ and $\frac{\ell_{\text{d-enc}}^\varepsilon(Y^n)}{n}$ such that the probability of error is less than ε .

Definition 5: (One-shot ε achievable rate pair) Let $(X, Y) \sim P_{XY}$, with range $\mathcal{X} \times \mathcal{Y}$. A one-shot rate pair (R_1, R_2) is called ε achievable if and only if there exists a $(1, 2^{\ell_{\text{d-enc}}^\varepsilon(X)}, 2^{\ell_{\text{d-enc}}^\varepsilon(Y)}, \varepsilon)$ one-shot code such that $\Pr\{X \neq \hat{X}\} \leq \varepsilon$, $\ell_{\text{d-enc}}^\varepsilon(X) \leq R_1$ and $\ell_{\text{d-enc}}^\varepsilon(Y) \leq R_2$.

Definition 6: (Asymptotically achievable rate pair) A rate pair (R_1, R_2) is asymptotically achievable if and only if there exists $(n, 2^{\ell_{\text{d-enc}}^\varepsilon(X^n)}, 2^{\ell_{\text{d-enc}}^\varepsilon(Y^n)}, \varepsilon)$ code such that $\Pr\{X^n \neq \hat{X}^n\} \leq \varepsilon$,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\ell_{\text{d-enc}}^\varepsilon(X^n)}{n} \leq R_1$$

and

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\ell_{\text{d-enc}}^\varepsilon(Y^n)}{n} \leq R_2.$$

Theorem 1: Let $(X, Y) \sim P_{XY}$, with range $\mathcal{X} \times \mathcal{Y}$. For the error $\varepsilon \in (0, 1)$. The following one-shot rate region for source coding of X with a helper observing Y is achievable

$$\begin{aligned}\ell_{\text{d-enc}}^\varepsilon(X) &\geq H_0^{\varepsilon_{11}}(X|U) - \log(\varepsilon - \varepsilon_{11}), \\ \ell_{\text{d-enc}}^\varepsilon(Y) &\geq D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y) \\ &\quad + \log[-\ln(\varepsilon_1 - \varepsilon_{11} - 2\varepsilon_{11}^{\frac{1}{2}})]\end{aligned}$$

for some conditional pmf $P_{U|Y}$, where $\varepsilon_1 < \varepsilon$ and ε_{11} is such that

$$\varepsilon_{11} + 2\varepsilon_{11}^{\frac{1}{2}} < \varepsilon_1 \text{ and } D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y) \geq 0. \quad (27)$$

Proof: The techniques used in the proof here are motivated from [13, Lemma 4.3]. Fix a conditional probability mass function $P_{U|Y}$ and let $P_U(u) = \sum_{y \in \mathcal{Y}} P_Y(y) P_{U|Y}(u|y)$. Choose ε_{11} such that the conditions in (27) are satisfied. Notice that such a choice of ε_{11} always exists because $D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y)$ is a decreasing function of ε_{11} . Let $Q \in \mathcal{B}^{\varepsilon_{11}}(P_{UX})$ and $\phi \in \mathcal{B}^{\varepsilon_{11}}(P_{UY})$ be such that

$$H_0^{\varepsilon_{11}}(X|U) = \log \max_{u \in \mathcal{U}} |\text{Supp}(Q(X|U=u))| \quad (28)$$

and

$$\begin{aligned}D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y) \\ = \log \max_{(u,y): P_{UY}(u,y) > 0} \log \frac{\phi(u,y)}{P_U(u)P_Y(y)},\end{aligned} \quad (29)$$

where

$$\phi(U=u|Y=y) := \begin{cases} \frac{\phi(u,y)}{P_Y(y)} & \text{if } P_Y(y) > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Notice that the triplet (X, Y, U) satisfy the following

$$X \rightarrow Y \rightarrow U. \quad (31)$$

For more details on (31) see [13, equation 4.4]. For every $(u, y) \in \mathcal{U} \times \mathcal{Y}$, let g be a mapping such that

$$g(u, y) := \sum_{x \in \mathcal{X}} P_{X|Y}(x|y) \mathbf{I}(x, u), \quad (32)$$

where $\mathbf{I}(x, u)$ for every $(x, u) \in \mathcal{X} \times \mathcal{U}$ is defined as follows

$$\mathbf{I}(x, u) = \begin{cases} 1 & \text{if } (x, u) \notin \text{Supp}(Q), \\ 0 & \text{otherwise.} \end{cases} \quad (33)$$

Define the following set

$$\mathcal{F} := \left\{ (u, y) \in \mathcal{U} \times \mathcal{Y} : g(u, y) \leq \varepsilon_{11}^{\frac{1}{2}} \right\}. \quad (34)$$

Random code generation: Randomly and independently assign an index $i \in [1 : 2^{\ell_{\text{d-enc}}^\varepsilon(X)}]$ to every realization $x \in \mathcal{X}$. The realizations with the same index

i form a bin $\mathcal{B}(i)$. Randomly and independently generate $2^{\ell_{\text{d-enc}}^\varepsilon(Y)}$ realizations $u(k)$, $k \in [1 : 2^{\ell_{\text{d-enc}}^\varepsilon(Y)}]$, each according to P_U .

Encoding: If the encoder 1 observes a realization $x \in \mathcal{B}(i)$, then the encoder 1 transmits i . For every realization $y \in \mathcal{Y}$ the encoder 2 finds an index k such that $(u(k), y) \in \mathcal{F}$. For the case when there are more than one such index, it sends the smallest one among them. If there is none, it then sends $k = 1$.

Decoding: The receiver finds the unique $x' \in \mathcal{B}(i)$ such that $(x', u(k)) \in \text{Supp}(Q)$.

Probability of error analysis: Let M_1 and M_2 be the chosen indices for encoding X and Y . The error in the above mentioned encoding decoding strategy occurs if and only if one or more of the following error events occur

$$\begin{aligned}E_1 &= \left\{ (U(m_2), Y) \notin \mathcal{F}, \forall m_2 \in [1 : 2^{\ell_{\text{d-enc}}^\varepsilon(Y)}] \right\}, \\ E_2 &= \{(X, U(M_2)) \notin \text{Supp}(Q)\}, \\ E_3 &= \{\exists x' \in \mathcal{B}(m_1) : (x', U(M_2)) \in \text{Supp}(Q), x' \neq X\}.\end{aligned}$$

For more details on error events see [17, Lemma 4]. The probability of error is upper bounded as follows

$$\Pr\{E\} \leq \Pr\{E_1\} + \Pr\{E_1^c \cap E_2\} + \Pr\{E_3|X \in \mathcal{B}(1)\}. \quad (35)$$

We now calculate $\Pr\{E_1\}$ as follows

$$\begin{aligned}\Pr\{E_1\} &= \sum_{y \in \mathcal{Y}} P_Y(y) \Pr \left\{ (U(m_2), y) \notin \mathcal{F}, \forall m_2 \in [1 : 2^{\ell_{\text{d-enc}}^\varepsilon(Y)}] \right\} \\ &\stackrel{a}{=} \sum_{y \in \mathcal{Y}} P_Y(y) \left(1 - \sum_{u: (u,y) \in \mathcal{F}} P_U(u) \right)^{2^{\ell_{\text{d-enc}}^\varepsilon(Y)}} \\ &\stackrel{b}{\leq} \sum_{y \in \mathcal{Y}} P_Y(y) \left(1 - 2^{-D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y)} \right)^{2^{\ell_{\text{d-enc}}^\varepsilon(Y)}} \\ &\quad \sum_{u: (u,y) \in \mathcal{F}} \phi(U=u|Y=y) \\ &\stackrel{c}{\leq} 1 - \sum_{(u,y) \in \mathcal{F}} \phi(u, y) + e^{-2^{\ell_{\text{d-enc}}^\varepsilon(Y)} 2^{-D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y)}} \\ &\stackrel{d}{\leq} \varepsilon_{11} + \Pr\{(U, Y) \notin \mathcal{F}\} + e^{-2^{\ell_{\text{d-enc}}^\varepsilon(Y)} 2^{-D_\infty^{\varepsilon_{11}}(P_{UY}||P_U \times P_Y)}},\end{aligned}$$

where a follows because $U(1), \dots, U(2^{\ell_{\text{d-enc}}^\varepsilon(Y)})$ are independent and subject to identical distribution P_U ; b follows from (7), (29) and (30); c follows from the inequalities $(1-x)^y \leq e^{-xy}$ ($0 \leq x \leq 1, y \geq 0$) and $e^{-xy} \leq 1-y+x$ ($x \geq 0, 0 \leq y \leq 1$) and (30); d is true

because of the following arguments

$$\begin{aligned}
1 - \varepsilon_{11} &\stackrel{a}{\leq} \sum_{(u,y) \in \mathcal{U} \times \mathcal{Y}} \phi(u, y) \\
&= \sum_{(u,y) \in \mathcal{F}^c} \phi(u, y) + \sum_{(u,y) \in \mathcal{F}} \phi(u, y) \\
&\stackrel{b}{\leq} \Pr\{\mathcal{F}^c\} + \sum_{(u,y) \in \mathcal{F}} \phi(u, y) \\
&\leq \Pr\{(U, Y) \notin \mathcal{F}\} + \sum_{(u,y) \in \mathcal{F}} \phi(u, y), \quad (36)
\end{aligned}$$

where a and b both follow from the fact that $\phi(u, y) \in \mathcal{B}^{\varepsilon_{11}}(P_{UY})$. By rearranging the terms in (36) we get

$$1 - \sum_{(u,y) \in \mathcal{F}} \phi(u, y) \leq \varepsilon_{11} + \Pr\{(U, Y) \notin \mathcal{F}\}. \quad (37)$$

We now calculate $\Pr\{(U, Y) \notin \mathcal{F}\}$ as follows

$$\begin{aligned}
\Pr\{(U, Y) \notin \mathcal{F}\} &= \Pr\{g(U, Y) \geq \varepsilon_{11}^{\frac{1}{2}}\} \\
&\stackrel{a}{\leq} \varepsilon_{11}^{-\frac{1}{2}} \mathbb{E}_{UY}(g(U, Y)) \\
&\leq \varepsilon_{11}^{-\frac{1}{2}} \sum_{(u,y) \in \mathcal{U} \times \mathcal{Y}} P_{UY}(u, y) g(u, y) \\
&\stackrel{b}{=} \varepsilon_{11}^{-\frac{1}{2}} \sum_{(u,y) \in \mathcal{U} \times \mathcal{Y}} P_{UY}(u, y) \\
&\quad \sum_{x \in \mathcal{X}} P_{X|Y}(x|y) \mathbf{I}(x, u) \\
&= \varepsilon_{11}^{-\frac{1}{2}} \sum_{\substack{(x,u,y) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y} \\ (u,x) \notin \text{Supp}(Q)}} P_{XUY}(x, u, y) \\
&= \varepsilon_{11}^{-\frac{1}{2}} \sum_{(u,x) \notin \text{Supp}(Q)} P_{UX}(u, x) \\
&\stackrel{c}{\leq} \varepsilon_{11}^{\frac{1}{2}}, \quad (38)
\end{aligned}$$

where a follows from Markov's inequality; b follows from (32) and c follows because of the following arguments

$$\begin{aligned}
1 - \varepsilon_{11} &\stackrel{a}{\leq} \sum_{(u,x) \in \mathcal{U} \times \mathcal{X}} Q(u, x) \\
&\stackrel{b}{\leq} \sum_{(u,x) \in \text{Supp}(Q)} P_{UX}(u, x), \quad (39)
\end{aligned}$$

where a follows from (28) and b follows because $Q(u, x) \leq P_{UX}(u, x)$, for every $(u, x) \in \mathcal{U} \times \mathcal{X}$. By rearranging the terms in (39) we get

$$1 - \sum_{(u,x) \in \text{Supp}(Q)} P_{UX}(u, x) \leq \varepsilon_{11}.$$

The second term in (35) is calculated as follows

$$\begin{aligned}
&\Pr\{E_1^c \cap E_2\} \\
&= \sum_{\substack{(x,u,y) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \\ (u,y) \in \mathcal{F}, (u,x) \notin \text{Supp}(Q)}} P_{XUY}(x, u, y) \\
&= \sum_{(u,y) \in \mathcal{F}} P_{UY}(u, y) \sum_{x: (x,u) \notin \text{Supp}(Q)} P_{X|Y}(x|y) \\
&\leq \varepsilon_{11}^{\frac{1}{2}}, \quad (40)
\end{aligned}$$

where the last inequality follows from (34). From (37), (38) and (40) it follows that

$$\begin{aligned}
&\Pr\{E_1\} + \Pr\{E_1^c \cap E_2\} \\
&\leq \varepsilon_{11} + 2\varepsilon_{11}^{\frac{1}{2}} + e^{-2^{\ell_{\text{d-enc}}^{\varepsilon}(Y)}} 2^{-D_{\infty}^{\varepsilon_{11}}(P_{UY} || P_U \times P_Y)}.
\end{aligned}$$

Let

$$\varepsilon_1 \geq \varepsilon_{11} + 2\varepsilon_{11}^{\frac{1}{2}} + e^{-2^{\ell_{\text{d-enc}}^{\varepsilon}(Y)}} 2^{-D_{\infty}^{\varepsilon_{11}}(P_{UY} || P_U \times P_Y)}. \quad (41)$$

It now easily follows that

$$\begin{aligned}
\ell_{\text{d-enc}}^{\varepsilon}(Y) &\geq \log[-\ln(\varepsilon_1 - \varepsilon_{11} - 2\varepsilon_{11}^{\frac{1}{2}})] \\
&\quad + D_{\infty}^{\varepsilon_{11}}(P_{UY} || P_U \times P_Y).
\end{aligned}$$

Finally, the third term in (35) is calculated as follows

$$\begin{aligned}
&\Pr\{E_3\} = \Pr\{E_3 | \mathcal{B}(1)\} \\
&= \sum_{(x,u) \in \mathcal{X} \times \mathcal{U}} \Pr\{(X, U) = (x, u) | X \in \mathcal{B}(1)\} \\
&\quad \Pr\left\{\exists x' \neq x : x' \in \mathcal{B}(1) \text{ and } Q(x', u) > 0 \right. \\
&\quad \left. | x \in \mathcal{B}(1), (X, U) = (x, u)\right\} \\
&\leq \sum_{(x,u) \in \mathcal{X} \times \mathcal{U}} P_{XU}(x, u) \sum_{\substack{x' \neq x \\ Q(x', u) > 0}} \Pr\{x' \in \mathcal{B}(1)\} \\
&\leq 2^{-\ell_{\text{d-enc}}^{\varepsilon}(X)} \sum_{(x,u) \in \mathcal{X} \times \mathcal{U}} P_{XU}(x, u) \max_{u \in \mathcal{U}} \sum_{x: Q(x, u) > 0} 1 \\
&\stackrel{a}{=} 2^{-\ell_{\text{d-enc}}^{\varepsilon}(X)} \sum_{(x,u) \in \mathcal{X} \times \mathcal{U}} P_{XU}(x, u) \max_{u \in \mathcal{U}} |\text{Supp}(Q(X|U = u))| \\
&= 2^{-\ell_{\text{d-enc}}^{\varepsilon}(X)} \max_{u \in \mathcal{U}} |\text{Supp}(Q(X|U = u))|, \quad (42)
\end{aligned}$$

where a follows because $Q(X = x|U = u) := \frac{Q(x, u)}{P_U(u)}$ and $Q(X = x|U = u) := 0$ if $P_U(u) = 0$. Thus from (41) and (42) it follows that

$$\Pr\{E\} \leq \varepsilon_1 + 2^{-\ell_{\text{d-enc}}^{\varepsilon}(X)} \max_{u \in \mathcal{U}} |\text{Supp}(Q(X|U = u))|.$$

Let

$$\varepsilon_1 + 2^{-\ell_{\text{d-enc}}^\varepsilon(X)} \max_{u \in \mathcal{U}} |\text{Supp}(Q(X|U=u))| \leq \varepsilon.$$

It now easily follows that

$$\ell_{\text{d-enc}}^\varepsilon(X) \geq H_0^{\varepsilon_1}(X|U) - \log(\varepsilon - \varepsilon_1).$$

This completes the proof. \blacksquare

The asymptotic optimality of the rate region obtained in Theorem 1 is an immediate consequence of Definition 6, Lemma 1 and 3.

V. CONCLUSION AND ACKNOWLEDGEMENTS

We proved that smooth max divergence and smooth max conditional Rényi entropy can be used to obtain one-shot achievable rate region for source coding when coded side information is available at the decoder. Furthermore, we showed that asymptotically this region coincides with the rate region as derived by Wyner in [13].

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